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On Wheeler–Feynman absorber theory of radiation

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Abstract. An attempt is made to re-derive the radiation damping formula using Wheeler–Feynman formalism of the absorber theory of radiation. The point of departure is that the absorber response on the radiating charged particle is calculated by taking into account the principle of action and reaction.

1. Introduction

It is well-known that Maxwell's equations are symmetrical in time and they admit both retarded as well as advanced solutions. An accelerated charge a radiates electromagnetic energy and it suffers radiation damping. This radiated energy would reach another charge b some distance away at a later instant. This corresponds to the retarded solution $F_{\text{ret}}^{(a)}$ of Maxwell's equations. The time-reversed situation, in which the energy converges on the charge from infinity at precisely that instant at which it accelerates, is never observed. This corresponds to the advanced solution $F_{\text{adv}}^{(a)}$ of Maxwell's equations. Of these, only the retarded solution is admitted, while the equally consistent advanced solution is rejected on the grounds of causality. However, Dirac (1938) proposed that, to obtain the empirically well-established formula for the radiation damping of an electron, it is necessary to employ both the retarded and the advanced fields. In his theory, the electron a is considered as a point charge and the expression $\frac{1}{2}(F_{\text{ret}}^{(a)} - F_{\text{adv}}^{(a)})$ is shown to be finite at every point including the location of the electron itself. This then correctly reproduces the radiation damping formula.

The presence of the advanced fields was also admitted by Wheeler and Feynman (1949—to be referred to as WF II) who developed a classical electrodynamics of direct interparticle action. They based their theory of action-at-a-distance electrodynamics on a single action principle due to Schwarzschild (1903), Tetrode (1922) and Fokker (1929). Their main results may be summarized as follows.

(i) An accelerated charge in an otherwise charge-free space does not radiate electromagnetic energy. Two charges interact through electromagnetic radiation only when they lie on each other's light cone. This is the principle of action and reaction.

(ii) The fields which act on a given charged particle arise only due to *other* charged particles. This eliminates the uncomfortable concept of self-action.

(iii) The field produced by a charged particle a is given by

$$F^{(a)} = \tfrac{1}{2}(F_{\text{ret}}^{(a)} + F_{\text{adv}}^{(a)}). \tag{1}$$

Wheeler and Feynman (1945—to be referred to as WF III), using the action-at-a-distance formalism, developed the suggestion by Tetrode, who proposed to abandon the concept of electromagnetic radiation as an elementary process and to interpret it as a consequence of an interaction between the source and all *other* charged particles in the universe, collectively called the absorber. Assuming that the universe contains

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enough charged particles to absorb completely the radiation emitted by the source, they derived the expression for the radiation damping for various situations. Furthermore, they demonstrated that both retarded and advanced solutions give equally consistent results and the explanation of the irreversibility of the radiation process observed in nature falls outside the domain of electrodynamics.

2. Modifications of Wheeler–Feynman arguments

Wheeler and Feynman (in WF III) consider a charged particle a, called the source, situated in a completely absorbing universe. This accelerates to send a fully retarded field outwards into the absorbing medium. The field that reaches an absorber particle b at a later instant is the *net* retarded field, which is the superposition of the proper field of the source a and the response of the absorber particles other than b. The returned field of the absorber particle b at the location of a is assumed to be 'elementary' travelling through the medium with the velocity of light in vacuum. The radiation of a fully retarded field by the source is justified on the grounds that the returned reaction of all the absorber particles evaluated at the source itself, when added to $F^{(a)}$ of the source, gives $F_{\text{ret}}{}^{(a)}$, which is in accord with experience. However, according to the standard procedure of the electromagnetic theory, the law of propagation of such a retarded disturbance through a dispersive medium must take into account the properties of the medium through its refractive index.

We wish to modify somewhat the arguments of Wheeler and Feynman in the following manner.

(i) Let the source particle a receive the acceleration at time t.

(ii) It radiates a fully retarded electromagnetic disturbance, which travels outwards.

(iii) The *net* retarded field disturbs the absorber particle b.

(iv) The absorber particle interacts back on the source through a *fully* advanced field, which is also a *net* field, hence not elementary.

(v) Summing over all the absorber particles $b \neq a$, the radiation reaction field is calculated.

The main point of departure from the original Wheeler–Feynman procedure is contained in step (iv) above, the justification for which is given as follows.

(*a*) The given absorber particle b receives the *net* retarded field which is the superposition of the proper field of the source a and those of the absorber particles other than b. By the principle of action and reaction, the response field of b should interact back with the particles other than b and the *net* field should reach the source a. This means that we should include the refractive index for the returned response field as well.

(*b*) It is shown in WF II that $F^{(a)}$ represents the field produced by a single particle. When a large number of particles are involved in the interaction, the field that will be experienced by any single charged particle, situated in the absorber, would be a superposition of the elementary fields of the form $F^{(a)}$.

(*c*) Since the absorber particle b itself would experience radiation damping when it radiates, it should respond by radiating a fully advanced field. This is the time-reversed version of the reason for which Wheeler and Feynman consider the source particle to radiate a fully-retarded field as in argument (ii) above.†

† We feel that this asymmetry is to be expected, as the problem itself is inherently asymmetric with respect to the source on the one hand and the absorber particles on the other.

(*d*) As the absorber particles may be free electrons as well as charged ions, they are assumed to have substantial inertia. Hence they are disturbed only after the 'principal part' of the disturbance from the source, namely the 'signal' in the sense of Brillouin (1960), arrives at the location of the particle. We assume, following Brillouin, that the 'signal' velocity ($<c$) is the same as the group velocity of the pulse, provided the frequencies emitted do not lie in the region of anomalous dispersion. Hence, if the source a, situated at the origin, accelerates at time t, the absorber particle b at a distance r_k from it, is disturbed at time $t+r_k/U$, where U is the group velocity of the pulse. On the other hand, if one treats the returned reaction as an elementary interaction travelling with the velocity of light c in vacuum, then the instant at which the returned reaction reaches the source is $(t+r_k/U-r_k/c) > t$, that is, it would reach the location of the source too late to account for the radiation damping, which occurs simultaneously with the acceleration of the source.

3. Derivation of the radiation damping formula

In this derivation, we assume that the absorber is composed of free charged particles which are either at rest or are moving slowly with respect to the source a. These charged particles are supposed to be well separated from each other, so that the medium may be considered to be of low density. The absorption of the radiation emitted by the source is assumed to be complete.

The source particle a is considered to have a charge $+e$. Let this source particle be accelerated at time t. Let us assume, following Wheeler and Feynman, that a fully retarded pulse is sent out from the source. The *net* field reaches the absorber particle b at a distance r_k from the source, at time $t+r_k/U$, where U is the group velocity of the pulse through the dispersive medium of the absorber. The absorber particle experiences an acceleration A_k given by

$$A_k = (e_k/m_k) \times \text{electric field due to the disturbance} \tag{2}$$

where e_k and m_k are the charge and the mass respectively, of the given absorber particle.

The electric field due to the disturbance has the magnitude $-(eA/r_k c^2)\sin\theta$, where θ is the angle between the acceleration A of the source and $\boldsymbol{r}_k$. Here, the electrostatic term, which vanishes as $1/r_k^2$, has been ignored.

The acceleration of the absorber particle is

$$A_k = -\frac{e_k}{m_k}\frac{eA}{r_k c^2}\sin\theta. \tag{3}$$

The force of radiation reaction on the source through the advanced returned field of the absorber particle, in the direction of the acceleration of the source, has the magnitude

$$-e\frac{e_k A_k}{r_k c^2}\cos\phi \tag{4}$$

where ϕ is the angle between the returned field and A. This reduces the expression (4) to

$$\frac{e^2 A}{c^4}\frac{e_k^2}{m_k r_k^2}\sin^2\theta.$$

To calculate the phase of the returned reaction, we represent the acceleration of the source as a Fourier integral

$$A = \int A_\omega \exp(-\mathrm{i}\omega t)\,\mathrm{d}\omega. \tag{5}$$

Now, if we subdivide the frequency interval into narrow regions of width $2\epsilon_l$, so that there is neither an overlap of these regions, nor are they disjoint, we get

$$A = \sum_{\omega_l} \int_{\omega_l-\epsilon_l}^{\omega_l+\epsilon_l} A_\omega \exp(-\mathrm{i}\omega t)\,\mathrm{d}\omega. \tag{6}$$

The acceleration of a typical absorber particle b at a distance r_k from the source, due to the interaction of the source on it, is

$$A_k = \sum_{\omega_l} \int_{\omega_l-\epsilon_l}^{\omega_l+\epsilon_l} A_{k\omega} \exp\{-\mathrm{i}(\omega t - \kappa r_k)\}\,\mathrm{d}\omega. \tag{7}$$

In each narrow region of ω, $\omega-\epsilon_l \leqslant \omega \leqslant \omega_l+\epsilon_l$, we expand $\kappa = \kappa(\omega) = \omega/V$, where V is the phase velocity of the disturbance for the frequency ω, in Taylor's series about ω_l, in the form

$$\kappa(\omega) = \frac{\omega}{V_l} + \left\{\frac{\partial(\omega/V)}{\partial\omega}\right\}_l (\omega-\omega_l) + \ldots \tag{8}$$

where we have written $\kappa(\omega_l) = \omega_l/V_l$. Ignoring second- and higher-order terms in $(\omega-\omega_l)$, we get

$$\kappa(\omega) = \frac{\omega_l}{V_l} + \left\{\frac{1}{V_l} - \frac{\omega_l}{V_l}\left(\frac{\mathrm{d}V}{\mathrm{d}\omega}\right)_l\right\}(\omega-\omega_l).$$

We obtain, by collecting terms,

$$\kappa(\omega) = \frac{\omega}{U_l} + \frac{\omega_l^2}{V_l^2}\left(\frac{\mathrm{d}V}{\mathrm{d}\omega}\right)_l$$

where $U_l = (1/V_l) - (\omega_l/V_l^2)(\mathrm{d}V/\mathrm{d}\omega)_l$ is the group velocity for the group whose frequencies lie in the neighbourhood of ω_l. The equation (7) then becomes

$$A_k = \sum_{\omega_l} \exp\left\{\mathrm{i}\frac{\omega_l^2}{V_l^2}\left(\frac{\mathrm{d}V}{\mathrm{d}\omega}\right)_l r_k\right\} \int_{\omega_l-\epsilon_l}^{\omega_l+\epsilon_l} A_{k\omega} \exp\left\{-\mathrm{i}\omega\left(t-\frac{r_k}{U_l}\right)\right\}\mathrm{d}\omega. \tag{9}$$

The advanced returned reaction from the absorber particle will produce an acceleration A_s in the source, given by

$$\begin{aligned} A_s &= \sum_{\omega_l} \exp\left\{\mathrm{i}\frac{\omega_l^2}{V_l^2}\left(\frac{\mathrm{d}V}{\mathrm{d}\omega}\right)_l r_k\right\} \int_{\omega_l-\epsilon_l}^{\omega_l+\epsilon_l} A_{s\omega} \exp\left\{-\mathrm{i}\omega\left(t-\frac{r_k}{U_l}+\frac{r_k}{U_l}\right)\right\}\mathrm{d}\omega \\ &= \sum_{\omega_l} \int_{\omega_l-\epsilon_l}^{\omega_l+\epsilon_l} A_{s\omega} \exp\left[-\mathrm{i}\left\{\omega t - \frac{\omega_l^2}{V_l^2}\left(\frac{\mathrm{d}V}{\mathrm{d}\omega}\right)_l r_k\right\}\right]\mathrm{d}\omega. \end{aligned} \tag{10}$$

If now the width of each frequency band is made sufficiently narrow, then for each frequency ω the phase difference between the returned reaction and the acceleration of the source can be written as

$$\frac{\omega^2}{V^2}\frac{\mathrm{d}V}{\mathrm{d}\omega}r_k. \tag{11}$$

For a medium of low density the refractive index corresponding to the electromagnetic disturbance of this frequency is given by

$$n = 1 - \frac{2\pi N e_k^{\,2}}{m_k \omega^2} \tag{12}$$

where N is the particle density of the absorber. Using $V = c/n$ and equation (12), we obtain

$$\frac{\omega^2}{V^2}\frac{\mathrm{d}V}{\mathrm{d}\omega}r_k = -\frac{4\pi N e_k^{\,2}}{m_k c \omega}r_k. \tag{13}$$

Hence, the force of radiation reaction on the source due to a typical absorber particle at the distance r_k, for the frequency ω, is

$$\frac{e^2 A}{c^4}\frac{e_k^{\,2}}{m_k r_k^{\,2}}\sin^2\theta \exp\left(-\frac{\mathrm{i}4\pi N e_k^{\,2} r_k}{m_k c\omega}\right).$$

Choosing the direction of $\boldsymbol{A}$ as the polar axis, we integrate the above expression for all the absorber particles, to give the total radiation reaction force on the source as

$$\frac{2e^2}{3c^3}A\omega\int_0^\infty \exp(-\mathrm{i}U)\,\mathrm{d}U = \frac{2e^2}{3c^3}A\omega(-\mathrm{i}) = \frac{2e^2}{3c^3}\frac{\mathrm{d}A}{\mathrm{d}t}. \tag{14}$$

This is the well-known radiation damping formula, which holds for the case of a slowly moving charged particle. It can be seen that this formula holds whatever may be the dependence of the acceleration upon time, so long as the velocities involved are non-relativistic.

4. Radiation reaction formula for the charged particle in a dense dispersive medium

We follow the same general scheme as in the first derivation. The source a is situated in an absorbing medium in which the particles of the absorber are no longer free nor are they far from each other. For such a medium, we may write the refractive index $(n-\mathrm{i}k)$ in the form

$$(n-\mathrm{i}k)^2 = 1 + \frac{4\pi N e_k^{\,2}/m_k}{\omega_0^{\,2}-\omega^2-2\mathrm{i}\beta\omega} = 1 - \frac{4\pi N e_k^{\,2}}{m_k\omega^2}p(\omega) \tag{15}$$

where $p(\omega) = -\omega^2/(\omega_0^{\,2}-\omega^2-2\mathrm{i}\beta\omega)$. The equation of motion for a charged particle, situated in a dispersive medium carrying an electromagnetic disturbance of frequency ω, is

$$m_k\ddot{\boldsymbol{r}}_k = -\frac{e\omega^2}{\omega_0^{\,2}-\omega^2-2\mathrm{i}\beta\omega}\boldsymbol{E}$$

or

$$\ddot{\boldsymbol{r}}_k = \frac{e_k}{m_k}p(\omega)\boldsymbol{E}.$$

Hence, the effective electric field acting on a charged particle situated in such a medium is $p(\omega)\boldsymbol{E}$.

We assume that a charged particle, called the source, is situated in such a medium and is accelerated at time t. We shall now evaluate the contribution of the absorber particles to the electric field in the vicinity of the source, at a distance r from it. The advanced field produced by the absorber at this point is obtained by integrating (over all the absorber particles) the product of the following factors.

(i) $A = A_0 \exp(-\mathrm{i}\omega t)$ is the Fourier component of the acceleration of the source, which, for simplicity, is assumed to be periodic.

(ii) $-(e/r_k c^2)\sin\theta$, where θ is the angle between $\boldsymbol{A}$ and $\boldsymbol{r}_k$, when multiplied by the acceleration of the source, gives the strength of the full retarded electric field in vacuum at a distance r_k from the source.

(iii) $\exp\{\mathrm{i}(\omega^2/v^2)(\mathrm{d}V/\mathrm{d}\omega)r_k\}$ gives the phase difference of the advanced field reacting on the source, relative to that of the source acceleration.

(iv) $\exp\{\mathrm{i}\omega r(n-\mathrm{i}k)\cos\phi/c\}$, where ϕ is the angle between $\boldsymbol{r}$ and $\boldsymbol{r}_k$, is the correction to be applied to the phase of the absorber field at the source itself, in order to evaluate this field at the distance r from the source.

(v) $(e_k/m_k)p(\omega)$ relates the acceleration of the absorber particle to the electric field experienced by it.

(vi) $p(\omega)$, this factor relates the returned *effective* advanced field to the electric field produced by the acceleration of the absorber particle.

(vii) $-(e_k/r_k c^2)\sin\theta$ times the acceleration of the absorber particle gives the magnitude of the component of the advanced field produced by the absorber particle, in the neighbourhood of the source and parallel to its acceleration.

(viii) $N r_k^2\,\mathrm{d}r_k\,\mathrm{d}\Omega$, this represents the number of absorber particles in the element of solid angle $\mathrm{d}\Omega$ and in the interval of distance $\mathrm{d}r_k$ at a distance r_k.

Thus, the total advanced field of the absorber in the direction of the acceleration of the source, at a small distance r from it, is given by

$$\frac{e}{c^3} A_0 \exp(-\mathrm{i}\omega t) \int_\Omega \exp\left\{\frac{\mathrm{i}\omega r(n-\mathrm{i}k)\cos\phi}{c}\right\} \sin^2\theta \frac{\mathrm{d}\Omega}{4\pi} \times \int_0^\infty \frac{4\pi N e_k^2}{m_k c} \{p(\omega)\}^2 \exp\left(\mathrm{i}\frac{\omega^2}{V^2}\frac{\mathrm{d}V}{\mathrm{d}\omega} r_k\right) \mathrm{d}r_k. \tag{16}$$

The first integral has been evaluated in WF III and its value is

$$\frac{2}{3}\left[F_0\left\{\frac{\omega r(n-\mathrm{i}k)}{c}\right\} - P_2(\cos\chi) F_2\left\{\frac{\omega r(n-\mathrm{i}k)}{c}\right\}\right] \tag{17}$$

where χ is the angle between $\boldsymbol{A}$ and $\boldsymbol{r}$ and

$$F_0(U) = \begin{cases} 1 & \text{for small } U \\ \dfrac{\{\exp(\mathrm{i}U) - \exp(-\mathrm{i}U)\}}{2\mathrm{i}U} & \text{for all } U \end{cases}$$

$$F_2(U) = \begin{cases} -U^2/15 & \text{for small } U \\ \dfrac{\{\exp(\mathrm{i}U) - \exp(-\mathrm{i}U)\}}{2\mathrm{i}U} & \text{for large } U. \end{cases}$$

We now evaluate the second integral. Using $V = c/(n-ik)$ and equation (15), a brief calculation yields

$$\frac{\omega^2}{V^2}\frac{dV}{d\omega} = \frac{1}{(n-ik)\omega}\left(\frac{4\pi N e_k^2}{m_k c}\right)\{p(\omega)\}^2 \tag{18}$$

provided the damping factor β is assumed small for the medium considered. The second integral over r_k can now be easily evaluated to give $-i(n-ik)\omega$.

Combining these results, we obtain that the advanced field of the absorber, near the source at a distance r from it, has an electric field component parallel to the acceleration of the source, given in magnitude and phase by the expression

$$\frac{2e}{3c^3}A_0\exp(-i\omega t)\{-i\omega(n-ik)\}\left[F_0\left\{\frac{\omega r(n-ik)}{c}\right\}-P_2(\cos\chi)F_2\left\{\frac{\omega r(n-ik)}{c}\right\}\right]. \tag{19}$$

This, at a distance of several wavelengths from the source, becomes

$$\left[-\frac{eA_0}{2rc^2}\exp\left\{-i\omega t+\frac{i\omega(n-ik)r}{c}\right\}+\frac{eA_0}{2rc^2}\exp\left\{-i\omega t-\frac{i\omega(n-ik)r}{c}\right\}\right]\sin^2\chi. \tag{20}$$

This is the Dirac radiation field $\frac{1}{2}(F_{\text{ret}}^{(a)}-F_{\text{adv}}^{(a)})$, which accounts for the radiation reaction of the point charge.

To evaluate the field at the location of the source itself, we imagine a small spherical region with the source at its centre, that contains no particle of the absorber. In this region $n = 1$, $k = 0$; also $F_0(U)_{U\to 0} \to 1$ and $F_2(U)_{U\to 0} \to 0\ (-U^2/15)$. The expression (18) then gives $(2e/3c^3)\ dA/dt$, which, when multiplied by the charge of the source, gives, again, the radiation damping formula.

5. Special cases

(i) A special case of the above should be noted. If the source is situated in a lossless dispersive medium, we can put $\beta = 0$ and this gives

$$n^2 = 1 - \frac{4\pi N e_k^2}{m_k\omega^2}p(\omega)$$

where

$$p(\omega) = -\frac{\omega^2}{\omega_0^2-\omega^2}.$$

We note that

(*a*) $\omega^2 < \omega_0^2$, $p(\omega) < 0$, $n > 1$ and (*b*) $\omega^2 > \omega_0^2$, $p(\omega) > 0$, $n < 1$.

One obtains for both these cases,

$$\left\{-\frac{eA_0}{2rc^2}\exp\left(-i\omega t+\frac{i\omega nr}{c}\right)+\frac{eA_0}{2rc^2}\exp\left(-i\omega t-\frac{i\omega nr}{c}\right)\right\}\sin^2\chi. \tag{21}$$

The derivations of (20) and (21) are valid only for the frequency regions (*a*) $\omega^2 < \omega_0^2$ and (*b*) $\omega^2 > \omega_0^2$, but break down when ω approaches the resonance frequency ω_0 from either side, as in this region the group velocity is no longer the 'signal' velocity.

(ii) We could also reproduce the derivation of (14) as a special case of the procedure outlined in § 4, by setting $p(\omega) = 1$ and using the refractive index given by (11).

Yet, we have given the derivation of (14) in detail to demonstrate the simplicity of the method and to illustrate the calculation of phase of the returned reaction.

6. Conclusions

We have shown that it is possible to derive the radiation damping formula for a point charge in the framework of the Wheeler–Feynman formalism of the absorber theory of radiation by modifying their cycle of arguments which assumes that the absorber response field is *elementary*. We feel in making this assumption that they have apparently not taken into account the principle of action and reaction which is one of the most important results of their action-at-a-distance electrodynamics.

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